Mathematical Finance
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Assignment 7

We fix throughout a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are given a filtration $\mathbb{F}$, unless otherwise stated.

A large financial market

We take here as a probability space $\Omega := [0,1]$, $\mathcal{F}$ the Borel–σ-algebra on $[0,1]$, and as probability measure $\mathbb{P}$ the Lebesgue measure on $[0,1]$. We consider then a financial market with time-horizon 1, and with countably many risky assets with (discounted) prices $(S^n)_{n\in\mathbb{N}}$ which are given for any $n \in \mathbb{N}$ by

$$S_t^n := 0,\ t \in [0,1),\ S_1^n(x) := \begin{cases} -x^{-1/2}, \text{ if } x \in [0,\varepsilon_n), \\ (1-x)^{-\frac{1}{n+1}}, \text{ if } x \in [\varepsilon_n, 1], \end{cases}$$

where the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ takes values in $(0,1)$, and converges to 0 as n goes to $+\infty$. We take for $\mathbb{F}$ the natural filtration generated by $(S^n)_{n\in\mathbb{N}}$.

1) Show that it is possible to choose the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that $\mathbb{E}^{\mathbb{P}}[S_1^n] = 1$ for all $n \in \mathbb{N}$.

2) We now want to prove that $\mathbb{P}$ is a separating measure for this market. Show that it is enough for this to prove that for any $n \in \mathbb{N}$ and any sequence $(c_k)_{k\in\{0,\dots,n\}}$ such that $\sum_{k=0}^n c_k S_1^k$ is bounded from below, we have

$$\mathbb{E}^{\mathbb{P}}\left[\sum_{k=0}^n c_k S_1^k\right] \leq 0,$$

and deduce that $\mathbb{P}$ is indeed a separating measure.

3) Prove that there cannot exist an equivalent σ-martingale measure on this market, and comment.

On separating measures

Consider a financial market where discounted prices are given by $S := (S^1, \dots, S_t^d)^\top_{t\in[0,T]}$ which is a d-dimensional $(\mathbb{F},\mathbb{P})$–semi-martingale and let $\mathbb{Q}$ be a measure equivalent to $\mathbb{P}$ on $\mathcal{F}_T$

1) Assume that $\mathcal{F}_0$ is trivial and that $\mathbb{Q}$ is a separating measure for S. Show that if S is $(\mathbb{F},\mathbb{P})$–locally bounded, then $\mathbb{Q}$ is an equivalent local martingale measure for S.

2) Assume that $\mathbb{Q}$ is an equivalent $(\mathbb{F},\mathbb{Q})$–σ-martingale measure for S. Show that it is also an equivalent separating measure.

3) Now assume that $d = 1$, that $(\mathcal{F}_t)_{t\in[0,T]}$ is the natural ($\mathbb{P}$-completed) filtration of S and that the process $S = (S_t)_{t\in[0,T]}$ is of the form

$$S_t = \begin{cases} 0, \text{ if } 0 \leq t < T, \\ X, \text{ if } t = T, \end{cases}$$

where X is normally distributed with mean $\mu \neq 0$ and variance $\sigma^2 > 0$ under $\mathbb{P}$. Show that in this case, the class $\mathcal{M}_{\text{sep}}(S,\mathbb{F},\mathbb{P})$ of equivalent separating measures for S is strictly bigger than $\mathcal{M}_\sigma(S,\mathbb{F},\mathbb{P})$.

Stop–loss start–gain strategy

Let the financial market on $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, $T < \infty$, be described by a reference asset $S^0 = 1$ and one risky asset S being a geometric Brownian motion, i.e.

$$\mathrm{d}S_t = S_t\big(\mu \mathrm{d}t + \sigma \mathrm{d}W_t\big),\ S_0 = s_0 > 0, \tag{0.1}$$

for some given constants $\mu \in \mathbb{R}$, $\sigma > 0$.

Fix $K > 0$. We start with one share if $S_0 > K$ and with no share if $S_0 \leq K$. Whenever the stock price falls below K (or equals K), the share is sold, and whenever the price returns to a level strictly above K, one share is bought again. Thus, the amount held in the reference asset is given by $\delta_t = -K\mathbf{1}_{\{S_t>K\}}$, $t \in [0,T]$, and the amount held in the risky asset is given by $\Delta_t = \mathbf{1}_{\{S_t>K\}}$, $t \in [0,T]$.

1) Verify that the geometric Brownian motion S satisfying (0.1) has the expression

$$S_t = s_0 \exp\big(\sigma W_t + (\mu - \sigma^2/2)t\big),\ t \in [0,T].$$

2) Show that for each $t \in (0,T]$, it holds that

$$\mathbb{P}[S_t > K] > 0,\ \text{and}\ \mathbb{P}[S_t < K] > 0.$$

3) Let $L^K(S)$ be the local time of S at K defined as in the lecture notes. Show that $\mathbb{P}[L^K_t(S) > 0] > 0$ holds for all $t \in (0,T]$.

   Hint: Recall that by Girsanov's theorem, there exists a measure $\mathbb{Q}$ which is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ and such that $(S_t)_{t\in[0,T]}$ is an $(\mathbb{F},\mathbb{Q})$–martingale. You can take the $\mathbb{Q}$-expectation of $(S_t - K)^+$ and apply Jensen's inequality to get the desired result. Tanaka's formula will be very helpful. You may also use the fact that if S is a continuous martingale and H is a bounded $\mathbb{F}$-predictable process, then the stochastic integral $\int_0^\cdot H\mathrm{d}S$ is also a continuous martingale.

4) Conclude that the so-called *stop–loss start–gain strategy* (δ, Δ) defined above is not a self-financing strategy.